AIP Journal of Applied Physics

Analytical Solutions to Eddy-Current Probe-Coil Problems

C. V. Dodd and W. E. Deeds

Citation: Journal of Applied Physics **39**, 2829 (1968); doi: 10.1063/1.1656680
View online: http://dx.doi.org/10.1063/1.1656680
View Table of Contents: http://scitation.aip.org/content/aip/journal/jap/39/6?ver=pdfcov
Published by the AIP Publishing

Analytical Solutions to Eddy-Current Probe-Coil Problems*

C. V. DODD
Metals and Ceramics Division, Oak Ridge National Laboratory, Oak Ridge, Tennessee
AND
W. E. DEEDS
Consultant from the University of Tennessee, Knoxville, Tennessee
(Received 28 September 1967; in final form 18 January 1968)

Solutions have been obtained for axially symmetric eddy-current problems in two configurations of wide applicability. In both cases, the eddy currents are assumed to be produced by a circular coil of rectangular cross section, driven by a constant amplitude alternating current. One solution is for a coil above a semi-infinite conducting slab with a plane surface, covered with a uniform layer of another conductor. This solution includes the special cases of a coil above a single infinite plane conductor or above a sheet of finite thickness, as well as the case of one metal clad on another. The other solution is for a coil surrounding an infinitely long circular conducting rod with a uniformly thick coating of another conductor. This includes the special cases of a coil around a conducting tube or rod, as well as one metal clad on a rod of another metal. The solutions are in the form of integrals of first-order Bessel functions giving the vector potential, from which the other electromagnetic quantities of interest can be obtained. The coil impedance has been calculated for the case of a coil above a two-conductor plane. The agreement between the calculated and experimental values is excellent.

I. INTRODUCTION

Electromagnetic problems are usually divided into three categories: low frequency, intermediate frequency, and high frequency. At low frequencies, static conditions are assumed; at high frequencies, wave equations are used. Both of these regions have been studied extensively. However, in the intermediate frequency range, where diffusion equations are used, very few problems have actually been solved. Eddy-current coil problems fall into this intermediate frequency region. This paper presents an accurate technique for analyzing the problems of eddy-current testing.

Eddy-current testing has been used in industry for many years. As early as 1879, Hughes[1] used an induction coil to sort metals. There have been numerous articles on the testing of materials with eddy currents. Some of the first papers dealing with both the theory and the practical aspects of eddy-current testing are by Förster,[2] Förster and Stambke,[3] and Förster.[4] In this series of papers, analyses are made of a coil above a conducting surface, assuming the coil to be a magnetic dipole, and of an infinite coil encircling an infinite rod. Hochschild[5] also gives an analysis of an infinite coil including some eddy-current distributions in the metal. Waidelich and Renken[6] made an analysis of the coil impedance using an image approach. Their theoretical results agreed well with theory for relatively high frequencies. Libby[7] presented a theory in which he assumed the coil was a transformer with a network tied to the secondary. This network representation gave good results when compared to experiment. The diffusion of eddy-current pulses (Atwood and Libby[8]) can be represented in this manner. Russell *et al.*[9] gave an analysis of a cup core coil where they assumed the flux was entirely coupled into the conductor. The semiempirical results agreed fairly well with the experimental measurements. Vein,[10] Cheng,[11] and Burrows[12] gave treatments based on delta-function coils, and Burrows continued with the development of an eddy-current flaw theory. Dodd and Deeds,[13] Dodd,[14] and Dodd[15] gave a relaxation theory to calculate the vector potential of a coil with a finite cross section. Here we extend a "closed-form" solution to such coils.

The vector potential is used as opposed to the electric and magnetic fields. The differential equations for the vector potential may be derived from Maxwell's equations, with the assumption of cylindrical symmetry. This differential equation will then be solved to obtain a "closed-form" solution.

For the "closed-form" solution, sinusoidal driving currents and linear, isotropic, and homogeneous media are assumed. Solutions will be obtained for two different conductor geometries: a rectangular cross-section coil above a plane with one conductor clad on another

* Research sponsored by the U.S. Atomic Energy Commission under contract with the Union Carbide Corporation.

[1] D. E. Hughes, Phil. Mag. **8**, 50 (1879).
[2] Friedrich Förster, Z. Metallk. **43**, 163 (1952).
[3] Friedrich Förster and Kurt Stambke, Z. Metallk. **45**, 166 (1954).
[4] Friedrich Förster, Z. Metallk. **45**, 197 (1954).
[5] R. Hochschild, *Progress in Nondestructive Testing* (The Macmillan Company, New York, 1959), Vol. 1.
[6] D. L. Waidelich and C. J. Renken, Proc. Natl. Electron. Conf. **12**, 188 (1956).
[7] H. L. Libby, *Broadband Electromagnetic Testing Methods*, Hanford Atomic Products Operation, HW-59614 (1959).
[8] K. W. Atwood and H. L. Libby, *Diffusion of Eddy Currents*, Hanford Atomic Products Operation, HW-79844 (1963).
[9] T. J. Russell, V. E. Schuster, and D. L. Waidelich, J. Electron. Control **13**, 232 (1962).
[10] P. R. Vein, J. Electron. Control **13**, 471 (1962).
[11] David H. S. Cheng, Ph.D. dissertation (University of Missouri, 1964).
[12] Michael Leonard Burrows, *A Theory of Eddy Current Flow Detection* (University Microfilms, Inc., Ann Arbor, Michigan, 1964).
[13] C. V. Dodd and W. E. Deeds, in *Proceedings of the Symposium on Physics and Nondestructive Testing* (Southwest Research Institute, San Antonio, Texas, 1963).
[14] C. V. Dodd, *A Solution to Electromagnetic Induction Problems*, Oak Ridge National Laboratory, ORNL-TM-1185 (1965); M. S. thesis, The University of Tennessee, 1965.
[15] C. V. Dodd, *Solutions to Electromagnetic Induction Problems*, Oak Ridge National Laboratory, ORNL-TM-1842 (1967); Ph.D. dissertation, the University of Tennessee, 1967.

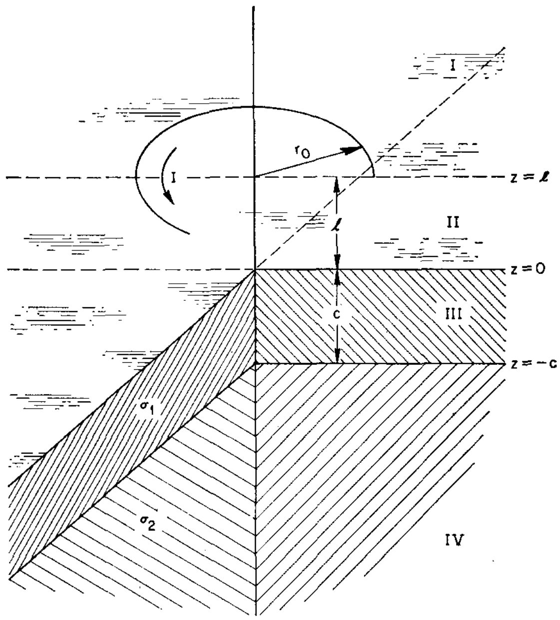

FIG. 1. Delta-function coil above a two-conductor plane.

and a rectangular cross-section coil encircling a two-conductor rod. The solutions for both geometries are given in terms of integrals of Bessel functions. Once the vector potential has been determined, it can be used to calculate any physically observable electromagnetic quantity.

Equations to calculate eddy-current density, induced voltage, coil impedance, and effect of defects will be given. Measured values of coil impedance as compared with calculated values show excellent agreement.

II. CLOSED-FORM SOLUTIONS OF THE VECTOR POTENTIAL

The differential equation for the vector potential, **A**, in an isotropic, linear, and inhomogeneous medium due to an applied current density $\mathbf{i}_0$ is[15]

$$\nabla^2\mathbf{A}=-\mu\mathbf{i}_0+\mu\sigma\partial\mathbf{A}/\partial t+\mu\epsilon\partial^2\mathbf{A}/\partial t^2+\mu\nabla(1/\mu)\times(\nabla\times\mathbf{A}). \tag{1}$$

The term $\mu\sigma\partial\mathbf{A}/\partial t$ is much greater than $\mu\epsilon\partial^2\mathbf{A}/\partial t^2$, so the latter may be neglected below about 10 MHz.

For most coil problems it is possible to assume axial symmetry as shown in Fig. 1. The vector potential will be symmetric about the axis of the coil. Since this assumption is valid for most problems and the alternative to this assumption is a much more complicated and impractical problem, axial symmetry is assumed. With axial symmetry, there is only a θ component of **I** and therefore of **A**. Expanding the θ component of Eq. (1) gives

$$\partial^2A/\partial r^2+(1/r)\partial A/\partial r+\partial^2A/\partial z^2-A/r^2=-\mu i_0+\mu\sigma\partial A/\partial t$$
$$-\mu\{\partial(1/\mu)/\partial r[(1/r)\partial rA/\partial r]+(\partial(1/\mu)/\partial z)\partial A/\partial z\}. \tag{2}$$

We assume that i_0 is a sinusoidal function of time, $i_0=i_0'e^{j\omega t}$. Then the vector potential is likewise a sinusoidal function of time,

$$A=A'\exp[j(\omega t+\phi)]=A''\exp(j\omega t).$$

Substituting into Eq. (2) gives

$$\frac{\partial^2A''}{\partial r^2}e^{j\omega t}+r^{-1}\frac{\partial A''}{\partial r}e^{j\omega t}+\frac{\partial^2A''}{\partial z^2}e^{j\omega t}-\frac{A''}{r^2}e^{j\omega t}=-\mu i_0'e^{j\omega t}+j\omega\mu\sigma A''e^{j\omega t}$$
$$-\mu\left\{\frac{\partial(1/\mu)}{\partial r}\left(r^{-1}\frac{\partial rA''}{\partial r}e^{j\omega t}\right)+\left[\frac{\partial(1/\mu)}{\partial z}\right]\frac{\partial A''}{\partial z}e^{j\omega t}\right\}.$$

Canceling out the term $e^{j\omega t}$ and dropping the prime gives

$$\frac{\partial^2A}{\partial r^2}+r^{-1}\frac{\partial A}{\partial r}+\frac{\partial^2A}{\partial z^2}-\frac{A}{r^2}=-\mu i_0+j\omega\mu\sigma A-\mu\left\{\frac{\partial(1/\mu)}{\partial r}\left(r^{-1}\frac{\partial rA}{\partial r}\right)+\left[\frac{\partial(1/\mu)}{\partial z}\right]\frac{\partial A}{\partial z}\right\}. \tag{3}$$

This is the general differential equation for the vector potential in a linear, inhomogeneous medium with a sinusoidal driving current. We shall now obtain a "closed-form" solution of Eq. (3).

We assume the medium to be linear, isotropic, and homogeneous. When I is the total driving current in a delta-function coil at (r_0, z_0), the general Eq. (3) then becomes:

$$\partial^2A/\partial r^2+(1/r)\partial A/\partial r+\partial^2A/\partial z^2-A/r^2-j\omega\mu\sigma A$$
$$+\mu I\delta(r-r_0)\delta(z-z_0)=0. \tag{4}$$

Once we have solved this linear differential equation for a particular conductor configuration, we can then superimpose any number of delta-function coils to build any desired shape of coil (provided that the current in each coil is known).

We solve the problem for two different conductor configurations: a coil above a two-conductor plane and a coil encircling a two-conductor rod. These two configurations apply to a large number of practical problems.

A. Coil Above a Two-Conductor Plane

The coil above a two-conductor plane is shown in Fig. 1. We have divided the problem into four regions. The differential equation in air (regions I and II) is

$$\partial^2A/\partial r^2+(1/r)\partial A/\partial r+\partial^2A/\partial z^2-A/r^2=0. \tag{5}$$

The differential equation in a conductor (regions

III and IV) is

$$\partial^2A/\partial r^2+(1/r)\partial A/\partial r+\partial^2A/\partial z^2-A/r^2-j\omega\mu\sigma_iA=0. \quad (6)$$

Setting $A(r, z)=R(r)Z(z)$ and dividing by $R(r)Z(z)$ gives

$$[1/R(r)]\partial^2R(r)/\partial r^2+[1/rR(r)]\partial R(r)/\partial r +[1/Z(z)]\partial^2Z(z)/\partial z^2-1/r^2-j\omega\mu\sigma_i=0. \quad (7)$$

We write for the z dependence

$$[1/Z(z)]\partial^2Z(z)/\partial z^2=\text{const}=\alpha^2+j\omega\mu\sigma_i, \quad (8)$$

or

$$Z(z)=A\exp(\alpha^2+j\omega\mu\sigma_i)^{1/2}z+B\exp-(\alpha^2+j\omega\mu\sigma_i)^{1/2}z. \quad (9)$$

We define

$$\alpha_i\equiv(\alpha^2+j\omega\mu\sigma_i)^{1/2}. \quad (10)$$

Equation (7) then becomes

$$[1/R(r)]\partial^2R(r)/\partial r^2+[1/rR(r)]\partial R(r)/\partial r +\alpha^2-1/r^2=0. \quad (11)$$

This is a first-order Bessel equation and has the solutions

$$R(r)=CJ_1(\alpha r)+DY_1(\alpha r). \quad (12)$$

Combining the solutions we have

$$A(r, z)=[A\exp(+\alpha_iz)+B\exp(-\alpha_iz)] \times[CJ_1(\alpha r)+DY_1(\alpha r)]. \quad (13)$$

We now need to determine the constants A, B, C, and D. They are functions of the separation "constant" α and are usually different for each value of α. Our complete solution would be a sum of all the individual solutions, if α were a discrete variable, but, since α is a continuous variable, the complete solution is an integral over the entire range of α. Thus, the general solution is

$$A(r, z)=\int_0^\infty[A(\alpha)\exp(+\alpha_iz)+B(\alpha) \times\exp(-\alpha_iz)][C(\alpha)J_1(\alpha r)+D(\alpha)Y_1(\alpha r)]d\alpha. \quad (14)$$

We must take $A(\alpha)=0$ in region I, where z goes to plus infinity. Due to the divergence of Y_1 at the origin, $D(\alpha)=0$ in all regions. In region IV, where z goes to minus infinity, $B(\alpha)$ must vanish. The solutions in each region then become

$$A^{(1)}(r, z)=\int_0^\infty B_1(\alpha)e^{-\alpha z}J_1(\alpha r)d\alpha \quad (15)$$

$$A^{(2)}(r, z)=\int_0^\infty[C_2(\alpha)e^{+\alpha z}+B_2(\alpha)e^{-\alpha z}]J_1(\alpha r)d\alpha \quad (16)$$

$$A^{(3)}(r, z)=\int_0^\infty[C_3(\alpha)\exp(\alpha_1z) +B_3(\alpha)\exp(-\alpha_1z)]J_1(\alpha r)d\alpha \quad (17)$$

$$A^{(4)}(r, z)=\int_0^\infty[C_4(\alpha)\exp(\alpha_2z)J_1(\alpha r)d\alpha. \quad (18)$$

The boundary conditions between the different regions are

$$A^{(1)}(r, l)=A^{(2)}(r, l) \quad (19)$$

$$(\partial/\partial z)A^{(1)}(r, z)\mid_{z=l}=(\partial/\partial z)A^{(2)}(r, z)\mid_{z=l}-\mu I\delta(r-r_0) \quad (20)$$

$$A^{(2)}(r, 0)=A^{(3)}(r, 0) \quad (21)$$

$$(\partial/\partial z)A^{(2)}(r, z)\mid_{z=0}=(\partial/\partial z)A^{(3)}(r, z)\mid_{z=0} \quad (22)$$

$$A^{(3)}(r, -c)=A^{(4)}(r, -c) \quad (23)$$

$$(\partial/\partial z)A^{(3)}(r, z)\mid_{z=-c}=(\partial/\partial z)A^{(4)}(r, z)\mid_{z=-c}. \quad (24)$$

Equation (19) gives

$$\int_0^\infty B_1(\alpha)e^{-\alpha l}J_1(\alpha r)d\alpha=\int_0^\infty[C_2(\alpha)e^{\alpha l} +B_2(\alpha)e^{-\alpha l}]J_1(\alpha r)d\alpha. \quad (25)$$

If we multiply both sides of Eq. (25) by

$$\int_0^\infty J_1(\alpha' r)rdr,$$

and then reverse the order of integration, we obtain

$$\int_0^\infty\frac{B_1(\alpha)e^{-\alpha l}}{\alpha}\left[\int_0^\infty J_1(\alpha r)J_1(\alpha' r)\alpha rdr\right]d\alpha =\int_0^\infty\alpha^{-1}[C_2(\alpha)e^{\alpha l}+B_2(\alpha)e^{-\alpha l}] \times\left[\int_0^\infty J_1(\alpha r)J_1(\alpha' r)\alpha rdr\right]d\alpha. \quad (26)$$

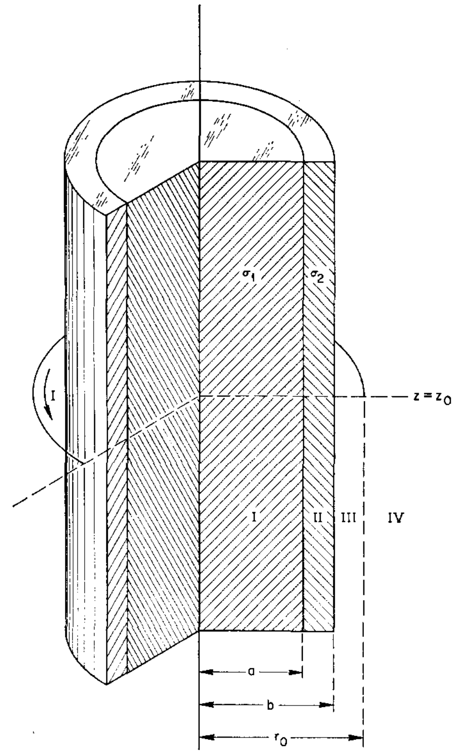

FIG. 2. Delta-function coil encircling a two-conductor rod.

We can simplify Eq. (26) by use of the Fourier–Bessel equation, which is

$$F(\alpha')=\int_0^\infty F(\alpha)\int_0^\infty J_1(\alpha r)J_1(\alpha' r)\alpha r dr d\alpha. \quad (27)$$

Equation (26) then becomes

$$(B_1/\alpha')e^{-\alpha' l}=(C_2/\alpha')e^{\alpha' l}+(B_2/\alpha')e^{-\alpha' l}. \quad (28)$$

We can evaluate the other integral equations in a similar manner. We get (after dropping the primes on the α):

$$-B_1e^{-\alpha l}=C_2e^{\alpha l}-B_2e^{-\alpha l}-\mu I r_0 J_1(\alpha r_0) \quad (29)$$

$$C_2/\alpha+B_2/\alpha=C_3/\alpha+B_3/\alpha \quad (30)$$

$$C_2-B_2=(\alpha_1/\alpha)C_3-(\alpha_1/\alpha)B_3 \quad (31)$$

$$(C_3/\alpha)\exp(-\alpha_1 c)+(B_3/\alpha)\exp(+\alpha_1 c)=(C_4/\alpha)\exp(-\alpha_2 c) \quad (32)$$

$$(\alpha_1/\alpha)C_3\exp(-\alpha_1 c)-(\alpha_1/\alpha)B_3\exp(\alpha_1 c)=(\alpha_2/\alpha)C_4\exp(-\alpha_2 c). \quad (33)$$

We now have six equations with six unknowns. Their solution is:

$$B_1=\tfrac{1}{2}\mu I r_0 J_1(\alpha r_0)\left\{e^{\alpha l}+\left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2)+(\alpha-\alpha_1)(\alpha_2+\alpha_1)e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2)+(\alpha+\alpha_1)(\alpha_2+\alpha_1)e^{2\alpha_1 c}}\right]e^{-\alpha l}\right\} \quad (34)$$

$$C_2=\tfrac{1}{2}\mu I r_0 J_1(\alpha r_0)e^{-\alpha l} \quad (35)$$

$$B_2=\tfrac{1}{2}\mu I r_0 J_1(\alpha r_0)\left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2)+(\alpha-\alpha_1)(\alpha_2+\alpha_1)e^{2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2)+(\alpha+\alpha_1)(\alpha_2+\alpha_1)e^{2\alpha_1 c}}\right]e^{-\alpha l} \quad (36)$$

$$C_3=\mu I r_0 J_1(\alpha r_0)\left[\frac{\alpha(\alpha_2+\alpha_1)e^{-\alpha l+2\alpha_1 c}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2)+(\alpha+\alpha_1)(\alpha_2+\alpha_1)e^{2\alpha_1 c}}\right] \quad (37)$$

$$B_3=\mu I r_0 J_1(\alpha r_0)\left[\frac{\alpha(\alpha_1-\alpha_2)e^{-\alpha l}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2)+(\alpha+\alpha_1)(\alpha_2+\alpha_1)e^{2\alpha_1 c}}\right] \quad (38)$$

$$C_4=\mu I r_0 J_1(\alpha r_0)\left[\frac{2\alpha_1\alpha e^{(\alpha_2+\alpha_1)c}e^{-\alpha l}}{(\alpha-\alpha_1)(\alpha_1-\alpha_2)+(\alpha+\alpha_1)(\alpha_2+\alpha_1)e^{2\alpha_1 c}}\right]. \quad (39)$$

We can now write the expressions for the vector potential in each region:

$$A^{(1)}(r,z)=\tfrac{1}{2}(\mu I r_0)\int_0^\infty J_1(\alpha r_0)J_1(\alpha r)\exp(-\alpha l-\alpha z)\times\left\{e^{2\alpha l}+\left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2)+(\alpha-\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)}{(\alpha-\alpha_1)(\alpha_1-\alpha_2)+(\alpha+\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)}\right]\right\}d\alpha \quad (40)$$

$$A^{(2)}(r,z)=\tfrac{1}{2}(\mu I r_0)\int_0^\infty J_1(\alpha r_0)J_1(\alpha r)e^{-\alpha l}\times\left\{e^{\alpha z}+\left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2)+(\alpha-\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)}{(\alpha-\alpha_1)(\alpha_1-\alpha_2)+(\alpha+\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)}\right]e^{-\alpha z}\right\}d\alpha. \quad (41)$$

$$A^{(3)}(r,z)=\mu I r_0\int_0^\infty J_1(\alpha r_0)J_1(\alpha r)e^{-\alpha l}\alpha\left[\frac{(\alpha_2+\alpha_1)\exp(2\alpha_1 c)\exp(\alpha_1 z)+(\alpha_1-\alpha_2)\exp(-\alpha_1 z)}{(\alpha-\alpha_1)(\alpha_1-\alpha_2)+(\alpha+\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)}\right]d\alpha \quad (42)$$

$$A^{(4)}(r,z)=\mu I r_0\int_0^\infty J_1(\alpha r_0)J_1(\alpha r)e^{-\alpha l}\alpha\left\{\frac{2\alpha_1\exp[(\alpha_2+\alpha_1)c]\exp(+\alpha_2 z)}{(\alpha-\alpha_1)(\alpha_1-\alpha_2)+(\alpha+\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)}\right\}d\alpha. \quad (43)$$

These are the equations for the vector potential of a delta-function coil above a two-conductor plane. Next we consider the derivation of the vector potential of a delta-function coil encircling a two-conductor rod.

B. Coil Encircling a Two-Conductor Rod

We assume a delta-function coil encircling an infinitely long, two-conductor rod, as shown in Fig. 2. The general differential equation is the same as Eq. (7) for a coil above a conducting plane.

$$[1/R(r)]\partial^2R(r)/\partial r^2+[1/rR(r)]\partial R(r)/\partial r+[1/Z(z)]\partial^2Z(z)/\partial z^2-1/r^2-j\omega\mu\sigma=0. \quad (44)$$

Now, however, we shall assume the separation constant to be negative:

$$[1/Z(z)]\partial^2Z(z)/\partial z^2=\text{“constant”}=-\alpha^2. \quad (45)$$

Then

$$Z(z)=F\sin\alpha(z-z_0)+G\cos\alpha(z-z_0), \quad (46)$$

and Eq. (44) becomes

$$r^2\partial^2R(r)/\partial r^2+r\partial R(r)/\partial r-[(\alpha^2+j\omega\mu\sigma)r^2+1]R(r)=0. \quad (47)$$

The solution to Eq. (47) in terms of modified Bessel functions is

$$R(r)=CI_1[(\alpha^2+j\omega\mu\sigma)^{1/2}r]+DK_1[(\alpha^2+j\omega\mu\sigma)^{1/2}r]. \tag{48}$$

We can now write the vector potential in each region. We use the fact that it is symmetric (with respect to $z-z_0$) to eliminate the sine terms, and the fact that $K_1(0)$ and $I_1(\infty)$ both diverge to eliminate their coefficients in regions I and IV, respectively. Thus we have:

$$A^{(1)}(r,z-z_0)=\int_0^\infty C_1(\alpha)I_1[(\alpha^2+j\omega\mu\sigma_1)^{1/2}r]\cos\alpha(z-z_0)\,d\alpha \tag{49}$$

$$A^{(2)}(r,z-z_0)=\int_0^\infty \{C_2(\alpha)I_1[(\alpha^2+j\omega\mu\sigma_2)^{1/2}r]+D_2(\alpha)K_1[(\alpha^2+j\omega\mu\sigma_2)^{1/2}r]\}\cos\alpha(z-z_0)\,d\alpha \tag{50}$$

$$A^{(3)}(r,z-z_0)=\int_0^\infty [C_3(\alpha)I_1(\alpha r)+D_3(\alpha)K_1(\alpha r)]\cos\alpha(z-z_0)\,d\alpha \tag{51}$$

$$A^{(4)}(r,z-z_0)=\int_0^\infty D_4(\alpha)K_1(\alpha r)\cos\alpha(z-z_0)\,d\alpha. \tag{52}$$

The boundary conditions between the different regions are

$$A^{(1)}(a,z-z_0)=A^{(2)}(a,z-z_0) \tag{53}$$

$$(\partial/\partial r)A^{(1)}(r,z-z_0)]_{r=a}=(\partial/\partial r)A^{(2)}(r,z-z_0)]_{r=a} \tag{54}$$

$$A^{(2)}(b,z-z_0)=A^{(3)}(b,z-z_0) \tag{55}$$

$$(\partial/\partial r)A^{(2)}(r,z-z_0)]_{r=b}=(\partial/\partial r)A^{(3)}(r,z-z_0)]_{r=b} \tag{56}$$

$$A^{(3)}(r_0,z-z_0)=A^{(4)}(r_0,z-z_0) \tag{57}$$

$$(\partial/\partial r)A^{(3)}(r,z-z_0)]_{r=r_0}=(\partial/\partial r)A^{(4)}(r,z-z_0)]_{r=r_0}+\mu I\delta(z-z_0). \tag{58}$$

If we multiply both sides of Eq. (53) by $\cos\alpha'(z-z_0)$ and integrate from zero to infinity, we obtain

$$\int_0^\infty\int_0^\infty C_1(\alpha)I_1[(\alpha^2+j\omega\mu\sigma_1)^{1/2}r]\cos\alpha(z-z_0)\cos\alpha'(z-z_0)\,d\alpha d(z-z_0)=\int_0^\infty\int_0^\infty [C_2(\alpha)I_1(\alpha^2+j\omega\mu\sigma_2)^{1/2}r]$$
$$+D_2(\alpha)K_1[(\alpha^2+j\omega\mu\sigma_2)^{1/2}r)][\cos\alpha(z-z_0)\cos\alpha'(z-z_0)(d\alpha d(z-z_0). \tag{59}$$

We can reverse the order of integration and use the orthogonality properties of the cosine integral or use the Fourier integral theorem:

$$\pi^{-1}\int_0^\infty f(\alpha)\left[\int_0^\infty \cos\alpha(z-z_0)\cos\alpha'(z-z_0)\,d(z-z_0)\right]d\alpha=f(\alpha'). \tag{60}$$

Thus, we can solve the integral Eqs. (53)–(58). We use α_1 and α_2 to designate $(\alpha^2+j\omega\mu\sigma_1)^{1/2}$ and $(\alpha^2+j\omega\mu\sigma_2)^{1/2}$.

After solving for the various constants, we get for the following equations for the vector potential in each region:

$$A^{(1)}(r,z-z_0)=\frac{\mu I}{\pi}\int_0^\infty \frac{r_0}{ab}\frac{K_1(\alpha r_0)}{D}I_1(\alpha_1 r)\cos\alpha(z-z_0)\,d\alpha \tag{61}$$

$$A^{(2)}(r,z-z_0)=\frac{\mu I}{\pi}\int_0^\infty \frac{r_0K_1(\alpha r_0)}{bD}\{[\alpha_2I_1(\alpha_1a)I_0(\alpha_2a)-\alpha_1I_1(\alpha_2a)I_0(\alpha_1a)]K_1(\alpha_2r)$$
$$+[\alpha_2K_0(\alpha_2a)I_1(\alpha_1a)+\alpha_1K_1(\alpha_2a)I_0(\alpha_1a)]I_1(\alpha_2r)\}\cos\alpha(z-z_0)\,d\alpha \tag{62}$$

$$A^{(3)}(r,z-z_0)=\frac{\mu I}{\pi}\int_0^\infty r_0K_1(\alpha r_0)\left(I_1(\alpha r)-\left\{\frac{K_1(\alpha_2b)}{bDK_1(\alpha b)}[\alpha_1I_1(\alpha_2a)I_0(\alpha_1a)-\alpha_2I_1(\alpha_1a)I_0(\alpha_2a)]\right.\right.$$
$$\left.\left.-\frac{I_1(\alpha_2b)}{bDK_1(\alpha b)}[\alpha_2K_0(\alpha_2a)I_1(\alpha_1a)+\alpha_1K_1(\alpha_2a)I_0(\alpha_1a)]+\frac{I_1(\alpha b)}{K_1(\alpha b)}\right\}K_1(\alpha r)\right)\cos\alpha(z-z_0)\,d\alpha \tag{63}$$

$$A^{(4)}=\frac{\mu I}{\pi}\int_0^\infty r_0K_1(\alpha r_0)K_1(\alpha r)\left\{\frac{K_1(\alpha_2b)[\alpha_2I_1(\alpha_1a)I_0(\alpha_2a)-\alpha_1I_1(\alpha_2a)I_0(\alpha_1a)]}{K_1(\alpha b)bD}+\frac{I_1(\alpha_2b)}{K_1(\alpha b)bD}[\alpha_2I_1(\alpha_1a)K_0(\alpha_2a)\right.$$
$$\left.+\alpha_1K_1(\alpha_2a)I_0(\alpha_1a)]-\frac{I_1(\alpha b)}{K_1(\alpha b)}+\frac{I_1(\alpha r_0)}{K_1(\alpha r_0)}\right\}\cos\alpha(z-z_0)\,d\alpha, \tag{64}$$

where we have defined

$$D \equiv [\alpha_2 K_0(\alpha_2 b) K_1(\alpha b) - \alpha K_0(\alpha b) K_1(\alpha_2 b)][\alpha_1 I_1(\alpha_2 a) I_0(\alpha_1 a) - \alpha_2 I_1(\alpha_1 a) I_0(\alpha_2 a)] + [\alpha_2 K_0(\alpha_2 a) I_1(\alpha_1 a) + \alpha_1 K_1(\alpha_2 a) I_0(\alpha_1 a)][\alpha I_1(\alpha_2 b) K_0(\alpha b) + \alpha_2 I_0(\alpha_2 b) K_1(\alpha b)]. \quad (65)$$

Equations (61)–(64) are the equations for the vector potential of a delta-function coil encircling a two-conductor rod. We shall now consider the superposition of the delta-function coils to form "real" coils.

C. Coils of Finite Cross Section

We have the equations for the vector potential produced by a single delta-function coil. We can now approximate any coil such as the ones shown in Figs. 3 and 4 by the superposition of a number of delta-function coils.

In general, we have

$$A(r,z)(\text{total}) = \sum_{i=1}^{n} A_i(r,z) = \sum_{i=1}^{n} A(r,z,l_i,r_i). \quad (66)$$

This equation is good for coils of any cross section. If we let the current distribution in the delta-function coils approach a continuous current distribution, we obtain

$$A(r,z)(\text{total}) = \int_{\text{coil cross section}} A(r,z,r_0,l)\, d(\text{area}), \quad (67)$$

where $A(r, z, l, r_0)$ is the vector potential produced by an applied current density $i_0(l, r_0)$. If the coil has a rectangular cross section, as in Figs. 3 and 4, we have

$$A(r,z)(\text{total}) = \int_{r_1}^{r_2} \int_{l_1}^{l_2} A(r,z,r_0,l)\, dr_0 dl. \quad (68)$$

We shall now assume that the applied current density $i_0(l, r_0)$ is a constant over the dimensions of the coil, that is, the current in each loop has the same phase and amplitude. We apply these results to Eq. (40), the case of a probe coil above a two-conductor plane.

After reversing the order of integration, we write

$$A^{(1)}(r,z) = \int_0^\infty \int_{r_1}^{r_2} \int_{l_1}^{l_2} \tfrac{1}{2}(\mu i_0 r_0) J_1(\alpha r_0) J_1(\alpha r) \exp[-\alpha(l+z)] \times \left\{ e^{2\alpha l} + \left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)} \right] \right\} d\alpha\, dr_0\, dl. \quad (69)$$

We express the integral over r_0 as

$$\int_{r_0=r_1}^{r_2} r_0 J_1(\alpha r_0)\, dr_0 = (1/\alpha^2) \int_{\alpha r_0=\alpha r_1}^{\alpha r_2} \alpha r_0 J_1(\alpha r_0)\, d\alpha r_0 = (1/\alpha^2) \int_{x=\alpha r_1}^{\alpha r_2} x J_1(x)\, dx \equiv (1/\alpha^2) I(r_2, r_1). \quad (70)$$

The integral over l is

$$\int_{l=l_1}^{l_2} \exp[-\alpha(l+z)](e^{2\alpha l}+1)\, dl = e^{-\alpha z} \int_{l=l_1}^{l_2} (e^{\alpha l} + e^{-\alpha l})\, dl = (e^{-\alpha z}/\alpha)\{[\exp(\alpha l_2) - \exp(\alpha l_1)] - [\exp(-\alpha l_2) - \exp(-\alpha l_1)]\}. \quad (71)$$

Upon applying Eqs. (70) and (71), the equations for the vector potential in the various regions for a rectangular cross-section coil become

$$A^{(1)}(r,z) = \tfrac{1}{2}(\mu i_0) \int_0^\infty (1/\alpha^3) I(r_2,r_1) J_1(\alpha r) e^{-\alpha z} \left\{ \exp(\alpha l_2) - \exp(\alpha l_1) - [\exp(-\alpha l_2) - \exp(-\alpha l_1)] \times \left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)} \right] \right\} d\alpha \quad (72)$$

$$A^{(2)}(r,z) = \tfrac{1}{2}(\mu i_0) \int_0^\infty (1/\alpha^3) I(r_2,r_1) J_1(\alpha r) [\exp(-\alpha l_1) - \exp(-\alpha l_2)] \times \left\{ e^{\alpha z} + \left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2) + (\alpha-\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)}{(\alpha-\alpha_1)(\alpha_1-\alpha_2) + (\alpha+\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)} \right] e^{-\alpha z} \right\} d\alpha \quad (73)$$

$$A^{(3)}(r,z)=\mu i_0\int_0^\infty (1/\alpha^3)I(r_2,r_1)J_1(\alpha r)[\exp(-\alpha l_1)-\exp(-\alpha l_2)]$$
$$\times\left[\frac{\alpha(\alpha_2+\alpha_1)\exp(2\alpha_1 c)\exp(\alpha_1 z)+\alpha(\alpha_1-\alpha_2)\exp(-\alpha_1 z)}{(\alpha-\alpha_1)(\alpha_1-\alpha_2)+(\alpha+\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)}\right]d\alpha \quad (74)$$

$$A^{(4)}(r,z)=\mu i_0\int_0^\infty (1/\alpha^3)I(r_2,r_1)J_1(\alpha r)[\exp(-\alpha l_1)-\exp(-\alpha l_2)]$$
$$\times\left\{\frac{2\alpha\alpha_1\exp[(\alpha_2+\alpha_1)c]\exp(\alpha_2 z)}{(\alpha-\alpha_1)(\alpha_1-\alpha_2)+(\alpha+\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)}\right\}d\alpha. \quad (75)$$

Equation (72) for $A^{(1)}$ is valid in the region above the coil and Eq. (73) for $A^{(2)}$ is valid for the region below the coil. We have to give special treatment to region I–II, between the top and bottom of the coil. For a point (r, z) in region I–II, we can use the equation $A^{(1)}(r, z)$ for the portion of the coil from z down to l_1 and the equation $A^{(2)}(r, z)$ for the portion of the coil from z up to l_2. If we substitute $l_2=z$ in Eq. (72) and $l_1=z$ in Eq. (73) and add the two equations, we get

$$A^{(1,2)}(r,z)=\tfrac{1}{2}(\mu i_0)\int_0^\infty (1/\alpha^3)I(r_2,r_1)J_1(\alpha r)\Big\{2-\exp[\alpha(z-l_2)]-\exp[-\alpha(z-l_1)]$$
$$+e^{-\alpha z}[\exp(-\alpha l_1)-\exp(-\alpha l_2)]\left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2)+(\alpha-\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)}{(\alpha-\alpha_1)(\alpha_1-\alpha_2)+(\alpha+\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)}\right]\Big\}d\alpha. \quad (76)$$

We now have the equations for the vector potential in all the regions.

III. CALCULATION OF PHYSICAL PHENOMENA

Once we have determined the vector potential, we can calculate any physically observable electromagnetic induction phenomenon. We shall now give the equations and perform the calculations for some of the phenomena of interest in eddy-current testing.

A. Induced Eddy Currents

We have, from Ohm's law:

$$\mathbf{J}=\sigma\mathbf{E}=-\sigma\partial\mathbf{A}/\partial t=-j\omega\sigma\mathbf{A}. \quad (77)$$

From the axial symmetry, Eq. (77) becomes

$$J=-j\omega\sigma A(r,z), \quad (78)$$

where $A(r, z)$ is given by either Eq. (74) or (75), depending on the region of interest.

B. Induced Voltage

We have, for the voltage induced in a length of wire

$$V=j\omega\int\mathbf{A}\cdot d\mathbf{s}. \quad (79)$$

For an axially symmetric coil with a single loop of radius r, Eq. (79) becomes

$$V=j\omega 2\pi r A(r,z). \quad (80)$$

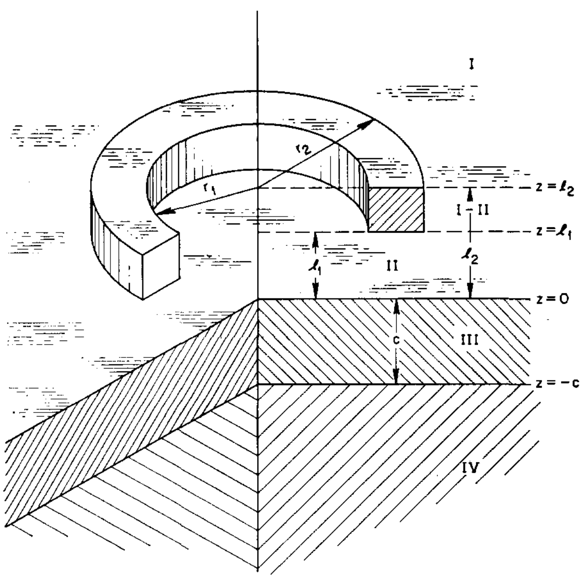

FIG. 3. Rectangular cross-section coil above a two-conductor plane.

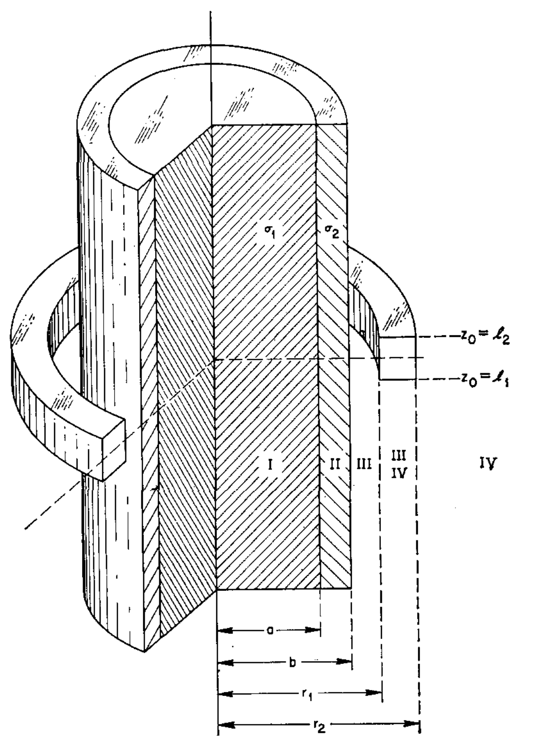

FIG. 4. Rectangular cross-section coil encircling a two-conductor rod.

The total voltage induced in a coil of n turns is then

$$V=j2\pi\omega\sum_{i=0}^{n} r_iA(r_i, z_i). \quad (81)$$

We can approximate the above summation by an integral over a turn density of N turns per unit cross-sectional area:

$$V\approx j2\pi\omega \iint_{\text{coil cross section}} rA(r,z)N\,dr\,dz. \quad (82)$$

For coils with a constant number of turns per unit cross-sectional area,

$$V=(j2\pi\omega n/\text{coil cross section}) \times \iint_{\text{coil cross section}} rA(r,z)\,dr\,dz. \quad (83)$$

This is the equation for the voltage induced in a coil by any coaxial coil.

When the two coils are one and the same, with cross-sectional area equal to $(l_2-l_1)(r_2-r_1)$, the self-induced voltage is

$$V=\frac{j2\pi\omega n}{(l_2-l_1)(r_2-r_1)}\int_{l_1}^{l_2}\int_{r_1}^{r_2} rA^{(1,2)}(r,z)\,dr\,dz. \quad (84)$$

C. Coil Impedance

From the self-induced voltage, we can calculate the coil impedance

$$V=ZI, \quad \text{or} \quad Z=V/I. \quad (85)$$

The current in a single loop is related to the applied current density, i_0, by

$$i_0=nI/(l_2-l_1)(r_2-r_1). \quad (86)$$

The coil impedance becomes

$$Z=\frac{j\omega\pi\mu n^2}{(l_2-l_1)^2(r_2-r_1)^2}\int_0^\infty \frac{1}{\alpha^5}I^2(r_2,r_1)\Bigg(2(l_2-l_1)+\alpha^{-1}\Bigg\{2\exp[-\alpha(l_2-l_1)]-2$$
$$+\{\exp(-2\alpha l_2)+\exp(-2\alpha l_1)-2\exp[-\alpha(l_2+l_1)]\}\left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2)+(\alpha-\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1c)}{(\alpha-\alpha_1)(\alpha_1-\alpha_2)+(\alpha+\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1c)}\right]\Bigg\}\Bigg)d\alpha. \quad (87)$$

This equation can be made more general by normalizing all the dimensions in terms of a mean coil radius $\bar{r}$.

$$\bar{r}=(r_1+r_2)/2. \quad (88)$$

All lengths are divided by $\bar{r}$ and all α's are multiplied by $\bar{r}$.

Upon normalization, Eq. (87) becomes

$$Z=\frac{j\omega\pi\mu n^2\bar{r}}{(l_2-l_1)^2(r_2-r_1)^2}\int_0^\infty \frac{1}{\alpha^5}I^2(r_2,r_1)\Bigg(2(l_2-l_1)+\alpha^{-1}\Bigg\{2\exp[-\alpha(l_2-l_1)]-2+\{\exp(-2\alpha l_2)+\exp(-2\alpha l_1)$$
$$-2\exp[-\alpha(l_2+l_1)]\}\left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2)+(\alpha-\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1c)}{(\alpha-\alpha_1)(\alpha_1-\alpha_2)+(\alpha+\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1c)}\right]\Bigg\}\Bigg)d\alpha. \quad (89)$$

The impedance may be normalized by dividing it by the magnitude of the air impedance. For the air impedance $\alpha_1=\alpha_2=\alpha$ and

$$Z_{\text{air}}=\frac{2\pi\omega\mu n^2\bar{r}}{(l_2-l_1)^2(r_2-r_1)^2}\int_0^\infty \frac{1}{\alpha^5}I^2(r_2,r_1)\left((l_2-l_1)+\alpha^{-1}\{\exp[-\alpha(l_2-l_1)]-1\}\right)d\alpha. \quad (90)$$

D. Flaw Impedance

Once the eddy-current density is known, we can simulate a flaw by superimposing a small current flowing in the opposite direction. The impedance change due to a small, spherical defect not too close to the surface (Burrows[12]) is

$$Z'=\tfrac{3}{2}\omega^2\sigma\,\text{vol}(A_{\text{defect}}/I)^2, \quad (91)$$

where A_{defect} is the vector potential at the defect, given by the equations for either $A^{(3)}$ and $A^{(4)}$ and "vol" is the volume of the defect.

E. Coil Inductance

The coil inductance is related to the magnitude of the air impedance by

$$\omega L=|Z_{\text{air}}| \quad (92)$$

or

$$L=\frac{2\pi\mu n^2\bar{r}}{(l_2-l_1)^2(r_2-r_1)^2}\int_0^\infty \frac{1}{\alpha^5}I^2(r_2,r_1)\left((l_2-l_1)+\alpha^{-1}\{\exp[-\alpha(l_2-l_1)]-1\}\right)d\alpha. \quad (93)$$

F. Mutual Inductance

The voltage generated in a "pickup" coil with dimensions r_2', r_1', l_2', l_1' by a current I flowing in a "driver" coil with dimensions r_2, r_1, l_2, l_1 is:

$$V = MdI/dt = j\omega MI, \tag{94}$$

or

$$M = V/j\omega I. \tag{95}$$

Using Eq. (83) to calculate the voltage we have

$$M = \frac{2\pi n'}{(\text{coil cross section})'} \iint_{(\text{coil cross section})'} rA(r,z)\,dr dz. \tag{96}$$

The equation for A will vary, depending on the region where the pickup coil is located. If the pickup coil is located in region I–II, the mutual inductance is

$$M = \frac{\mu\pi nn'\bar{r}}{(l_2'-l_1')(r_2'-r_1')(l_2-l_1)(r_2-r_1)} \int_0^\infty \frac{1}{\alpha^5} I(r_2,r_1) I(r_2',r_1') \Bigg(2(l_2'-l_1')$$
$$+\alpha^{-1}\Big\{\exp[-\alpha(l_2-l_1')]-\exp[-\alpha(l_2-l_2')]+\exp[-\alpha(l_2'-l_1)]-\exp[-\alpha(l_1'-l_1)]+\{\exp[-\alpha(l_2'+l_2)]$$
$$-\exp[-\alpha(l_2'+l_1)]-\exp[-\alpha(l_1'+l_2)]+\exp[-\alpha(l_1'+l_1)]\}\left[\frac{(\alpha+\alpha_1)(\alpha_1-\alpha_2)+(\alpha-\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)}{(\alpha-\alpha_1)(\alpha_1-\alpha_2)+(\alpha+\alpha_1)(\alpha_2+\alpha_1)\exp(2\alpha_1 c)}\right]\Big\}\Bigg) d\alpha. \tag{97}$$

This is the mutual inductance between the driver coil and the pickup coil in the presence of a clad conductor. By the reciprocity theorem, this is equal to the mutual inductance between the pickup coil and the driver coil.

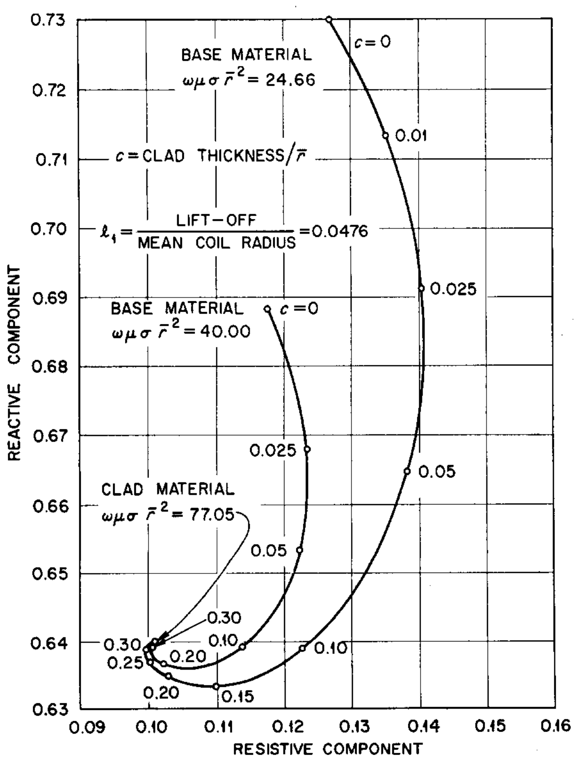

FIG. 5. Variation of normalized impedance with clad thickness.

G. Evaluation of Integrals

The normalized impedance has been calculated using a C–E–I–R time-sharing computer to evaluate integral Eqs. (89) and (90). The solutions have been programmed for any rectangular coil dimensions and liftoff (coil-to-conductor spacing) as well as for a metal of any conductivity clad (in varying thickness) onto a base metal of any conductivity. The programs, in "BASIC" language, and their descriptions have been reported.[16]

Figure 5 shows how the normalized impedance varies as a function of clad thickness.

IV. EXPERIMENTAL VERIFICATION

A family of four coils was constructed with different mean radii but all with the same normalized dimensions. The coil impedance was measured at various values of normalized liftoff and at various values of $\bar{r}^2\omega\mu\sigma$. The values of the experimental normalized coil impedance and the calculated normalized coil impedance are plotted in Fig. 6. The agreement between the calculated and measured values is excellent at the higher frequencies. At the lower frequencies, the measurements are very difficult to make, and the accuracy of the measured values becomes very poor. (Because of this, few eddy-current tests are made at these frequencies.) Thus, the theory is in excellent agreement with experimental values at the frequencies of interest in eddy-current testing.

[16] C. V. Dodd and W. E. Deeds, *Analytical Solutions to Eddy-Current Probe Coil Problems*, Oak Ridge National Laboratory, ORNL-TM-1987 (1967).

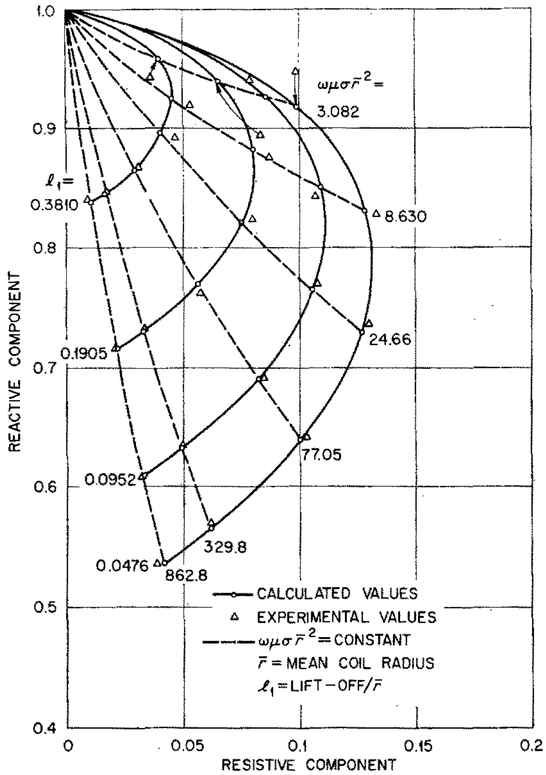

FIG. 6. Variation of experimental and calculated values of normalized impedance with frequency and liftoff.

V. ACCURACY OF CALCULATIONS

This technique, like most others used in engineering, is "exact, except for a few assumptions we have to make in order to work the problem." We shall now discuss the probable errors in some of these assumptions.

A. Axial Symmetry

This is a very good assumption, but we cannot easily wind coils that have perfect axial symmetry. This error will vary with the winding technique and will decrease as the number of turns on the coil and the coil-to-conductor spacing increases. This error will be effectively reduced when normalized impedance is calculated. For a typical coil, it should be less than 0.01%.

B. Current Sheet Approximation

This error arises because we have assumed a current sheet, while we actually have a coil wound with round, insulated wire. Some correction formulas are given by Rosa and Grover[17] for the inductance of a coil in air. From Eqs. (87) and (93) by Rosa and Grover, we have calculated the following correction formula:

$$\Delta L/L=\{[0.5058r_2-0.2742r_1+0.44(l_2-l_1)]/n\}\times(\ln D/d+0.155), \quad (98)$$

where all dimensions are normalized by the mean coil radius. The symbols D and d are the wire diameters with and without insulation, respectively. For a typical coil with 100 turns, the change in inductance is 0.19%. The change in normalized impedance will be a small fraction of the change in inductance.

C. High-Frequency Effects

These are probably the most serious sources of error in this calculation technique. As the frequency increases, the current density ceases to be uniformly distributed over the cross section of the wire, but becomes concentrated near the surface. The resistance of the coil increases, and the inductance decreases. The current is capacitively coupled between the turns in the coil, tending to flow across the loops of wire, rather than through them. Both the interwinding capacitance and the coil-to-metal sample capacitance increase. The coil-to-sample capacitance can be reduced by winding the coil such that the turns nearest the sample are electrically near alternating-current ground. The coil-to-sample capacitance will be much less than the interwinding capacitance. If the coil is used at frequencies where the interwinding capacitance has a small effect, the error in calculated normalized impedance will be a much smaller effect.

VI. CONCLUSIONS

This technique presents a quick and easy way to calculate the observed effects of actual eddy-current tests to a high degree of accuracy.

ACKNOWLEDGMENTS

The authors wish to express their appreciation to W. A. Simpson for performing the experimental measurements, to J. W. Luquire, and W. G. Spoeri for editing and checking the equations and their assistance in programming the stepwise solution of the integral equations.

[17] Edward B. Rosa and Frederick W. Grover, Natl. Bur. Stds. (U.S.), Tech. News Bull. **8**, 1 (1912).